Arithmetic and pseudo-arithmetic billiards

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Abstract

The arithmetic triangular billiards are classically chaotic but have Poissonian energy level statistics, in ostensible violation of the BGS conjecture. We show that the length spectra of their periodic orbits divides into subspectra differing by the parity of the number of reflections from the triangle sides; in the quantum treatment that parity defines the reflection phase of the orbit contribution to the Gutzwiller formula for the energy level density. We apply these results to all 85 arithmetic triangles and establish the boundary conditions under which the quantum billiard is "genuinely arithmetic", i. e., has Poissonian level statistics; otherwise the billiard is "pseudo-arithmetic" and belongs to the GOE universality class

1 Introduction

Quantum systems whose classical counterpart is chaotic, have level statistics belonging to one of the Wigner-Dyson universality classes of the random matrix theory; classically regular systems with more than one degree of freedom have Poissonian level statistics [1]. This statement, originally the "BGS conjecture"[2, 3], is now well established on the basis of the semiclassical approximation [4, 5, 6, 7] and some essential properties of the chaotic motion like orbit bunching [8]. The so called arithmetic systems represent thus a paradox; they have completely chaotic classical limit but level statistics close to Poissonian [9, 10, 11, 12, 13, 14].

Here we concentrate on the triangular arithmetic billiards [15, 16, 17]. They are formed by geodesics on the Riemann surface of constant negative curvature and have angles $\pi/l, \pi/m, \pi/n$ where l, m, n are certain integers or infinity; there exist 85 arithmetic triangles. Consecutively reflecting the triangle in its sides we can tesselate the complete Riemann surface; the group $T^*(l, m, n)$ generated by the reflections is the symmetry group of the Hamiltonian of a particle moving on

the tesselated surface. Due to that symmetry, there exists an infinite amount of the so called Hecke integrals of motion such that the level repulsion associated with Wigner-Dyson level statistics is absent and Poissonian statistics arises.

In a more naive approach we can solve the Schrödinger equation for the particle in an isolated triangle imposing Dirichlet or Neumann boundary conditions on its sides. As demonstrated numerically, only certain combinations of boundary conditions lead to the spectra with Poissonian level statistics [17]; we shall then speak of a "genuinely arithmetic", or simply arithmetic, quantum billiard as opposed to the "pseudo-arithmetic" ones with Wigner-Dyson levels statistics of the GOE class. E. g., if one of the angles is an odd fraction of π, the triangle can be genuinely arithmetic only if the boundary conditions on the adjacent sides of that angle are the same [17, 16]. In a problem following from desymmetrization of the solution on the tesselated Riemann surface, only such boundary conditions occurs.

To completely make peace with BGS, one has to show that the semiclassical treatment of the arithmetic systems indicates indeed Poissonian statistics. At a first glance it looks straightforward. A convenient measure of the spectral statistics is the form factor $K(\tau)$ obtained by the Fourier transformation of the level-level correlation function [1]. The diagonal approximation [4] applicable at least at small times, gives $K(\tau) \approx g(\tau T_H)\tau$ where T_H is the Heisenberg time and $g(T)$ is the average action multiplicity of periodic orbits with period T. In a "normal" system of the GOE universality class $g = 2$; on the other hand, in the arithmetic systems $g(T) \propto e^{\lambda T/2}/\lambda T$ where λ is the Lyapunov constant. Therefore $g(\tau T_H)$ grows to infinity in the limit $\hbar \to 0$ when $T_H \to \infty$ such that the form factor rises almost vertically from zero at $\tau = 0_+$ to values comparable with 1, as it should be for systems with Poissonian level statistics.

What remained unanswered for some time was why some triangles are genuinely arithmetic and others only pseudo-arithmetic, since the boundary conditions don't influence the periodic orbit set. The reason turned out to be the reflection phase with which the periodic orbit contributes to the Gutzwiller formula for the level density [18]. Only if the reflection phase is the same for all orbits comprising every length multiplet, their contributions to the form factor interfere constructively and the form factor does behave like $g(\tau T_H)\tau$. As shown for the triangle $(2, 3, 8)$, the possibility of the constructive interference is connected with the division of the periodic orbit multiplets into several classes differing by parities of the number of reflections from the triangle sides [16, 18].

In the present paper we extend the investigation to all 85 arithmetic triangles establishing the orbit length spectra subdivisions and the arithmetic boundary conditions. Our reasoning will not be much more complicated than, say, "$q_0 + q_1\sqrt{2}$ with rational $q_{0/1}$ cannot be equal to $\sqrt{3}\left(q_0' + q_1'\sqrt{2}\right)$ with rational $q_{0/1}'$ unless all q's are zero". We show that three scenarios are realized:

- The reflection phase of all orbits within a length multiplet is the same regardless of the boundary conditions, hence the quantum triangle is always genuinely arithmetic. This is the rarest case found in only four right

triangles $(2, m, n)$ when m and n are both even and not multiples of each other;

- The triangle is genuinely arithmetic if certain two sides of the triangle are both Dirichlet or both Neumann. Observed in 51 triangles;

- The triangle is genuinely arithmetic if the boundary conditions on all sides are the same. That scenario is realized in the equilateral triangles and in triangles whose two angles are odd fractions of π or zero. Observed in 30 triangles.

2 Preliminaries

2.1 Gutzwiller level density, reflection phase, form factor

The semiclassical theory of the quantum spectral statistics is based on the Gutzwiller trace formula for the level density [19]; for a billiard on pseudosphere it can be written,

$$\rho\left(E\right) \propto \sum_{\gamma} A_\gamma \exp\left(i\frac{S_\gamma}{\hbar} + i\Phi_\gamma\right)$$

where γ are periodic orbits with the action S_γ and the stability coefficients A_γ. The phase Φ_γ takes into account the phase gain π after every reflection from the sides with the Dirichlet boundary condition,

$$\Phi_\gamma = \nu_L^{(\gamma)}\phi_L + \nu_M^{(\gamma)}\phi_M + \nu_N^{(\gamma)}\phi_N.$$

Here $\nu_{L/M/N}^{(\gamma)}$ is the number of visits of the side L, M or N by the orbit γ and $\phi_{L/M/N}$ is 0 (or π) if the boundary condition on the side is Neumann (Dirichlet). Substituting the Gutzwiller density into the two-point correlation function we obtain, after the Fourier transform, the form factor as a double sum over orbits with the period in the interval $t, t+\Delta$ where Δ is some small time interval,

$$K\left(\tau\right) \propto \frac{1}{\Delta} \sum_{t < T_\gamma < t+\Delta} A_\gamma A_{\gamma'} \exp\left[\frac{i\left(S_\gamma - S_{\gamma'}\right)}{\hbar} + i\left(\Phi_\gamma - \Phi_{\gamma'}\right)\right], \quad t = \tau T_H.$$

Berry's diagonal approximation neglects pairs of orbits with different actions, or which is the same in a billiard, with different lengths. Denoting the multiplets of orbits with the same length by Λ we can write the form factor of the diagonal approximation as

$$K_{\text{diag}}\left(\tau\right) \propto \sum_{\Lambda} A_\Lambda^2 \sum_{\gamma, \gamma' \in \Lambda} \left(\pm 1\right) \tag{1}$$

where $\left(\pm 1\right) = \exp\left[i\left(\Phi_\gamma - \Phi_{\gamma'}\right)\right]$. In a "normal" GOE-system we would have $\gamma' = \gamma$ or $\gamma' = \left(\gamma\right)_{\text{TR}}$ ("TR"=time reversed); both γ and $\left(\gamma\right)_{\text{TR}}$ have the

same number of bumps against any side such that their reflection phases always cancel. In the arithmetic systems the numbers of visits $\nu_{L/M/N}^{(\gamma)}$ vary greatly within the length multiplets and needn't coincide for γ', γ . It is not obvious why the proclaimed enhancement of the form factor due to the pathologically high action/length multiplicity is not destroyed by cancelation of (± 1) in the inner sum. To put it bluntly, how can the energy spectrum of an arithmetic triangle be Poissonian unless all its sides are Neumann?

The question was answered in [18] which used the triangle $(2, 3, 8)$ as example. It was shown that all orbits belonging to a length multiplet have the same parity of the number of reflections ν_M from the side opposite to the angle $\pi/3$ and the same parity of the total number of reflections $\nu_L + \nu_N$ from its adjacent sides; the parity of ν_L and ν_N separately is not fixed. Therefore if the boundary condition on L and N is the same, the reflection phases of γ, γ' belonging to the same length multiplet cancel such that all summands in the inner sum in (1) are 1. The form factor then indeed is given by $g(\tau T_H)\tau$ at small τ , i. e., is Poissonian-like, and the triangle is genuinely arithmetic. In a pseudo-arithmetic $(2, 3, 8)$ the orbit pairs other than the GOE ones, make mutually cancelling contributions such that the Wigner-Dyson $K_{\text{diag}}(\tau) \sim 2\tau$ is restored.

Below we make a similar purely classical investigation of all 85 arithmetic triangles.

2.2 Some mathematical reminders

- An *algebraic number η of degree n* is the root of a polynomial with integer coefficients whose minimal power is n; that polynomial is called the *minimal polynomial* of η.

- A *field $\mathcal{K}$* is a set of objects closed with respect to addition, subtraction, multiplication and division by a non-zero. Dropping division we would define a *ring*. Example: rational numbers form the field Q; usual integers form a ring.

- An *algebraic field $Q(\eta)$* where η is algebraic, is an extension of Q consisting of the results of the field operations on the binomials $q_0 + q_1\eta$ with q_0, q_1 rational; the *degree* of $Q(\eta)$ is the power of the minimal polynomial of η. Any element of the field $q(\eta) \in Q(\eta)$ can be uniquely represented by a polynomial,
$$q(\eta) = q_0 + q_1\eta + \ldots + q_{n-1}\eta^{n-1}$$
where $q_0, \ldots, q_{n-1}$ are rationals, n is the degree of η.

- An *algebraic field $Q(\eta, \zeta)$ generated by two algebraic numbers* consists of binomials $q_0 + q_1\eta$, $\quad q_0' + q_1'\zeta$ with rational coefficients, their products and sums of the products. E.g., $Q(\cos \pi/4, \cos \pi/6)$ consists of the numbers $q(\sqrt{2}, \sqrt{3}) = q_0 + q_1\sqrt{2} + q_2\sqrt{3} + q_3\sqrt{6}$ and is equivalent to the field $Q(\cos \pi/12)$.

It will often be needed to know whether a certain algebraic number ζ belongs to the field $Q(\eta)$; in most cases the answer will be obvious. Note a useful relation concerning the algebraic numbers $\cos \pi/m$ where m is integer,

$$\cos \frac{\pi}{m} \in Q\left(\cos \frac{2\pi}{m}\right), \quad m \text{ odd};$$

$$\cos \frac{\pi}{m} \notin Q\left(\cos \frac{2\pi}{m}\right), \quad m \text{ even}. \tag{2}$$

The algebraic computer subroutines like "ToNumberField$[\zeta, \eta]$" of Wolfram Mathematica can be helpful in less transparent situations.

2.3 Möbius transformations in Poincaré plane

Classical periodic orbits in the arithmetic triangular billiards consist of pieces of geodesics separated by specular reflections from the triangle sides. The motion on the Riemann surface is conveniently mapped to the Poincaré complex half-plane $\text{Im}\, z > 0$ where the geodesics are depicted either by circles with the center on the real axis or by straight lines parallel to the imaginary axis [13]. After the reflection $\hat{\Sigma}$ in the triangle side the points z of a geodesic are transformed into the points z' of the mirror-reflected geodesic by the complex conjugation followed by Möbius transformation, $z' = \frac{a\bar{z}+b}{c\bar{z}-a}$. The matrix of reflection

$$\Sigma = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

is defined up to a sign; it has real elements and $\det \Sigma = -1$. The reflection operation must not be mixed with its matrix; e. g., $\hat{\Sigma}^2$ is identity but Σ^2 can be $+I$ or $-I$. The product of two reflections in the sides adjacent to the angle α is equivalent to rotation by 2α about the crossing point. Transformation of the geodesics after such a rotation is Möbius transformation whose matrix is a product of the reflection matrices; its determinant is 1 and its trace coincides with $2\cos\alpha$, up to a sign,

Any periodic orbit in a triangle can be encoded by the list of its ν consecutive reflections; the trace of the product W of the respective reflection matrices ("the orbit trace" for short) is connected with the orbit length l by,

$$2\cosh \frac{l}{2} = |\text{Tr}\, W|, \quad \nu \text{ even};$$

$$2\sinh \frac{l}{2} = |\text{Tr}\, W|, \quad \nu \text{ odd}.$$

Many different orbits can have the same trace which leads to the length multiplet formation. On the other hand, two orbits, γ_1 with an even number of reflections, and γ_2 with an odd one, can have equal length only if

$$(\text{Tr}\, W_{\gamma_1})^2 - (\text{Tr}\, W_{\gamma_2})^2 = 4. \tag{3}$$

This is an additional constraint which either excludes formation of the length multiplets with mixed parity of ν or reduces their number to a proportion exponentially small in the limit of large orbit lengths. Complete investigation of the exceptional situations when (3) is fulfilled, can be found for the triangle $(2, 3, 8)$ in the archived version of [18]. Since our interest lies in the quantum level statistics which is not influenced by the exceptional orbits, we do not extend this investigation to other arithmetic triangles. Numerical simulations confirm that the multiplets with mixed parity of ν are indeed either extremely rare or absent in all arithmetic triangles. An immediate physical consequence is that all arithmetic triangles with the Dirichlet boundary conditions on every side, are genuinely arithmetic.

2.4 Approximate length degeneracy of orbits with even and odd number of reflections

Unlike exact coincidence, systematic approximate equality of lengths l_e, l_o of the orbits with even (e) and odd (o) number of reflections ν does occur in some billiards. Namely, it happens when the corresponding traces coincide exactly, $\mathrm{Tr}\, W_e = \mathrm{Tr}\, W_o$; the necessary condition of the $eo-$trace degeneracy is given below in the end of Section 3.3. The orbit lengths exist then in doublets with the spacing $\Delta l_{oe} = l_o - l_e \approx 4e^{-l}$ which is to be compared with the much greater spacing between the length of orbits in the absence of the trace degeneracy, $\Delta l_{av} = \mathrm{const} \times e^{-l/2}$. The well-known example is Artin's billiard $(2, 3, \infty)$; a short stretch of its length spectrum around $l = 8$ is shown in Fig.1a. For comparison, we show a similar plot of the lengths of orbits of the billiard $(2, 3, 8)$, see Fig.1b, in which the $eo-$trace degeneracy is forbidden by the arithmetic considerations.
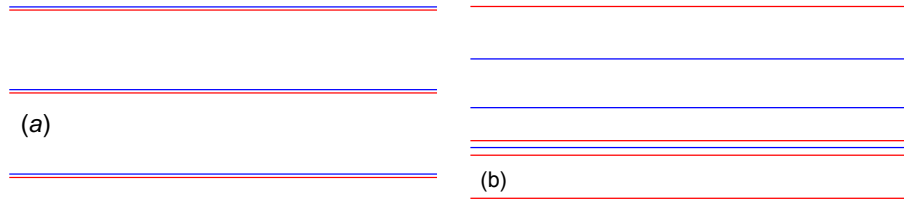

Figure 1: Stretch of PO length spectra of a) Artin's billiard $(2, 3, \infty)$ and b) the billiard $(2, 3, 8)$. Lengths of orbits with even and odd number of reflections are shown by blue and red lines respectively

The importance of the approximate eo length degeneracy becomes obvious when we consider the off-diagonal contributions $\propto \cos \frac{\Delta S}{\hbar}$ of the $eo-$doublets

to the form factor $K(\tau)$. It is easy to show that in the semiclassical regime the phase difference $\Delta S/\hbar$ is much larger than 1 when the time is smaller than the Ehrenfest time $\tau_E = T_E/T_H$. For $\tau > \tau_E$ the phase difference exponentially fast tends to zero such that the *eo*-splitting can be neglected; the diagonal approximation must therefore to be reformulated. In the crossover region around τ_E the behavior of $K(\tau)$ is expected to be complicated; cf. the exact $K(\tau)$ for Artin's billiard obtained in [20].

The "normal" off-diagonal contributions connected with the off-diagonal orbit pairs with non-coinciding $\operatorname{Tr} W$ become significant at times $\tau \gtrsim 2\tau_E$; systematic contributions of that type are expected to arise from the pairs of orbit-partners consisting of approximately the same pieces traversed in a different order and, perhaps, with different sense[8].

3 Method

3.1 Reflection matrices; original and mirror triangles

All but 5 arithmetic triangles are right or can be reduced to the right ones with non-equal legs by desymmetrization. Therefore we start with the triangles $(2, m, n)$, $m \neq n$, such that L is the hypotenuse of the triangle and N, M are its two legs. The vertices formed by the crossing of LM, MN, LN will be denoted O, Q and P; the respective angles will be $\alpha = \frac{\pi}{n}$, $\pi/2$ and $\beta = \frac{\pi}{m}$. It will be convenient to direct the side M along the imaginary axis and choose $z_Q = i$; then the two other vertices will be

$$z_O = i\frac{\cos\beta + \rho}{\sin\alpha}, \quad z_P = \frac{\rho + i\sin\beta}{\cos\alpha},$$
$$\rho \equiv \sqrt{\cos^2\beta + \cos^2\alpha - 1}.$$

The Möbius matrices of reflections in the sides M, N, L are

$$\Sigma_M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma_N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_L(\alpha, \beta) = \begin{pmatrix} -\cos\alpha & \cos\beta + \rho \\ \cos\beta - \rho & \cos\alpha \end{pmatrix}. \tag{4}$$

Their products describe rotations by $\pi, 2\beta$ and 2α about Q, P and O,

$$R_Q = \Sigma_M \Sigma_N = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5}$$

$$R_P(\alpha, \beta) = \Sigma_N \Sigma_L = \begin{pmatrix} \cos\beta - \rho & \cos\alpha \\ -\cos\alpha & \cos\beta + \rho \end{pmatrix},$$

$$R_O(\alpha, \beta) = \Sigma_L \Sigma_M = \begin{pmatrix} -\cos\alpha & -\cos\beta - \rho \\ \cos\beta - \rho & -\cos\alpha \end{pmatrix}.$$

The code $\mathbb{R}$ of the orbit with an even number of reflections can be written as a sequence of the elementary rotations. The trace of the corresponding orbit matrix R belongs to a ring with integer coefficients generated by
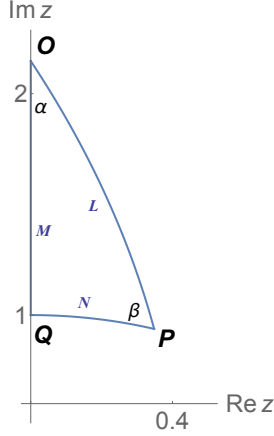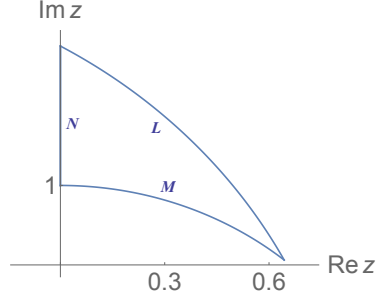
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Figure 2: Right triangle in Poincaré half-plane

Figure 3: Same triangle reflected in the bisector of the right angle

$2\cos\pi/l,\ 2\cos\pi/m,\ 2\cos\pi/n$, [15]; in fact in all cases one of the generators can be dropped because it is either integer (case $l = 2, 3, \infty$) or coincides with one of the other two generators (symmetric triangles). For our purposes a weaker statement is sufficient that the traces are elements of the field

$$\mathrm{Tr}\, R \in Q\left(\cos\pi/m, \cos\pi/n\right) \equiv K_T. \tag{6}$$

We shall show that in the majority of the arithmetic triangles this field can be divided into four or two sets which have no common elements except zero; membership in a particular set is defined by parity of the number of elementary rotations composing $\mathbb{R}$.

Simultaneously with $(2, m, n)$ we consider its mirror twin $(2, n, m)$ directingt its leg N opposite to π/n along the imaginary axis in the Poincaré half-plane (Fig. 2). The corresponding periodic orbits in the two triangles are built of the same sequences of reflections $\hat{\Sigma}_L, \hat{\Sigma}_M, \hat{\Sigma}_N$ in the sides opposite to the angles $\pi/l, \pi/m, \pi/n$; the related Möbius transformations will be given by the matrices $\Sigma_L\left(\beta, \alpha\right), \Sigma_N, \Sigma_M$ introduced above. Therefore that "mirror approach" amounts to the substitutions $\alpha \leftrightarrows \beta, M \leftrightarrows N$ in the matrix products.

3.2 Even number of reflections: alternatives for the matrix traces

The matrix of an orbit with even number of reflections is a rotation matrix and, in the representation we have chosen, can be one of the two types. The first one is the Fuchsian matrix [15, 13, 14] whose general form is,

$$R^{(I)} = \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2\sqrt{b} + x_3\sqrt{ab} \\ x_2\sqrt{b} - x_3\sqrt{ab} & x_0 - x_1\sqrt{a} \end{pmatrix} \tag{7}$$

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where $x_0, \ldots, x_3, a > 0, b > 0$ are real numbers belonging to some algebraic field K; the determinant of the matrix is 1. The Fuchsian matrices form a group: the product of two matrices $R_1^{(I)} R_2^{(I)}$ with the same K, a, b but different x is also Fuchsian with the same K, a, b. Our elementary rotation matrix R_P is Fuchsian with

$$K = Q\left(\cos^2 \alpha, \cos \beta\right) = Q\left(\cos 2\alpha, \cos \beta\right); \tag{8}$$
$$a = \cos^2 \beta + \cos^2 \alpha - 1 = \rho^2, \quad b = \cos^2 \alpha / \rho^2$$

and $x_0 = \cos \beta, \quad x_1 = -1, \quad x_2 = 0, \quad x_3 = 1$; consequently any power of R_P is also of the type $R^{(I)}$ with K, a, b given in (7). The field K is either a subfield of the field K_T defined in (6) or coincides with it.

If we multiply $R^{(I)}$ by the elementary rotation matrix R_Q we get a different-looking creature,

$$R^{(II)} = R^{(I)} R_Q = \begin{pmatrix} -x_2\sqrt{b} - x_3\sqrt{ab} & x_0 + x_1\sqrt{a} \\ -x_0 + x_1\sqrt{a} & x_2\sqrt{b} - x_3\sqrt{ab} \end{pmatrix}. \tag{9}$$

The left multiplication of $R^{(II)}$ also produces a matrix of the type $R^{(II)}$ with $x_1 \to -x_1, x_2 \to -x_2$. Multiplication of $R^{(II)}$ by R_Q produces $R^{(I)}$; in other words R_Q toggles the matrix type between I and II. The multiplication table for the types can be symbolically written,

$$I \times I = I, \quad II \times II = I, \quad II \times I = II, \quad I \times II = II. \tag{10}$$

The elementary rotation R_O is of the type II with $x_0 = -\cos \beta, x_1 = -1, x_2 = 0, x_3 = -1$ such that multiplication of the orbit matrix by R_O, same as by R_Q, toggles the matrix type between I and II. Consequently, if the code consists of an even total number of the elementary rotations $R_O = \Sigma_M \Sigma_L$ and $R_Q = \Sigma_M \Sigma_N$ coinciding with the number of reflections ν_M from the side M, its matrix will belong to the type $R^{(I)}$, otherwise it is $R^{(II)}$. Since the total number of reflections ν is assumed even we can just as well say that the type is $R^{(I)}$ if $\nu_N + \nu_L = \nu - \nu_M$ is even, and $R^{(II)}$ if it is odd.

A matrix $R^{(II)}$ could be $R^{(I)}$ in disguise differing only by a similarity transformation. However we can compare the corresponding traces which are basis independent,
$$\operatorname{Tr} R^{(I)} = 2x_0, \quad \operatorname{Tr} R^{(II)} = -2x_3\sqrt{ab}.$$

Whereas $\operatorname{Tr} R^{(I)}$ belongs to the field $K = Q\left(\cos 2\alpha, \cos \beta\right)$, the trace $\operatorname{Tr} R^{(II)}$ does so if and only if $\sqrt{ab} = \cos \alpha$ is contained in K; that field coincides then with the field K_T. If it doesn't, K is a subfield of K_T and the "coset"[1] $\cos \alpha\, K$ has no non-zero common elements with K. The traces of $R^{(I)}$ and $R^{(II)}$ cannot then be equal, consequently the corresponding periodic orbits cannot have equal lengths.

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[1]The field contains zero and is therefore not a group with respect to multiplication, hence "coset" in quotation marks

We can repeat these arguments for the mirror-reflected triangle and get an alternative division of the rotation matrices into two types. The first one is given by the Fuchsian matrices $R^{(I')}$ with

$$K' = Q\left(\cos\alpha, \cos 2\beta\right),$$
$$a' = \cos^2\beta - \sin^2\alpha, \quad \sqrt{a'b'} = \cos\beta;$$

the elementary rotation matrix $R_O = \Sigma_M\Sigma_L$ and all its powers are of the type $R^{(I')}$. Multiplication by R_Q and R_P changes the matrix type to II' analogous to (9) but with K, a, b replaced by K', a', b'; the multiplication table for the primed types is similar to (10). Consequently the rotation matrix is of the type I' if its code contains even total number of the elementary rotations $R_Q = \Sigma_M\Sigma_N$ and $R_P = \Sigma_N\Sigma_L$ coinciding with the number of reflections ν_N from the side N; otherwise its type is II'. The corresponding traces look like

$$\operatorname{Tr} R^{(I')} = 2x'_0, \quad \operatorname{Tr} R^{(II')} = -2x'_3\sqrt{a'b'}$$

where x', a', b' belong to K' as well as $\operatorname{Tr} R^{(I')}$. The trace $\operatorname{Tr} R^{(II')}$ will belongs to K' if and only if $\cos\beta \in K'$; if it doesn't the orbits with ν_N even and odd cannot have coinciding lengths.

Combining the two approaches we get the following alternatives: The fields $Q\left(\cos 2\alpha, \cos\beta\right)$ and $Q\left(\cos\alpha, \cos 2\beta\right)$ can coincide (e.g., if $\alpha = \beta$) or not; $\cos\alpha$ (resp. $\cos\beta$) can belong to $Q\left(\cos 2\alpha, \cos\beta\right)$ (resp. $Q\left(\cos\alpha, \cos 2\beta\right)$) or not. That gives at most four algebraic types of traces and correspondingly four types of the length multiplets of periodic orbits with even number of reflections.

3.3 Orbits with odd number of reflections

In the first approach we write the code with an odd number of symbols as $\hat{\Sigma}_M\hat{\mathbb{R}}$ where $\hat{\mathbb{R}}$ is a product of rotations, and the corresponding Möbius matrix as $W = \Sigma_M R$; note that $\operatorname{Tr} W = R_{11} - R_{22}$. The rotational body R contains $\nu_M - 1$ reflections in M and looks like (7) or (8) depending on parity of $\nu_M - 1$,

$$\operatorname{Tr} W = 2x_1\sqrt{a} \in \rho Q\left(\cos 2\alpha, \cos\beta\right), \quad \nu_M - 1 \quad \text{even,}$$
$$\operatorname{Tr} W = -2x_2\sqrt{b} \in \rho\cos\alpha\, Q\left(\cos 2\alpha, \cos\beta\right), \quad \nu_M - 1 \quad \text{odd;}$$

we remind that $\rho = \sqrt{\cos^2\beta + \cos^2\alpha - 1}$. In the mirror approach we write the code as $\hat{W} = \hat{\Sigma}_N\hat{\mathbb{R}}$ and the corresponding matrix as $W = \Sigma'_M R$ where R' is obtained by the substitutions $\hat{\Sigma}_L \to \Sigma_L\left(\beta, \alpha\right), \quad \hat{\Sigma}_{M/N} \to \Sigma_{N/M}$; the orbit trace is again the difference of the diagonal elements of $\mathbb{R}'$. Considering that $\rho\left(\alpha, \beta\right) = \rho\left(\beta, \alpha\right)$ we get

$$\operatorname{Tr} W \in \rho Q\left(\cos\alpha, \cos 2\beta\right), \quad \nu_N - 1 \quad \text{even;}$$
$$\operatorname{Tr} W \in \rho\cos\beta\, Q\left(\cos\alpha, \cos 2\beta\right), \quad \nu_N - 1 \quad \text{odd.}$$

α	β	K	K'	ρ
$\frac{\pi}{6}$	$\frac{\pi}{4}$	$Q\left(\sqrt{2}\right)$	$Q\left(\sqrt{3}\right)$	$\frac{1}{2}$
$\frac{\pi}{10}$	$\frac{\pi}{4}$	$Q\left(\sqrt{5},\sqrt{2}\right)$	$Q\left(\cos\frac{\pi}{10}\right)$	$\sqrt{\frac{\sqrt{5}+1}{8}}$
$\frac{\pi}{18}$	$\frac{\pi}{4}$	$Q\left(\cos\frac{\pi}{9}\right)+\sqrt{2}Q\left(\cos\frac{\pi}{9}\right)$	$Q\left(\cos\frac{\pi}{18}\right)$	$\sqrt{\frac{1}{2}\cos\frac{\pi}{9}}$
$\frac{\pi}{8}$	$\frac{\pi}{6}$	$Q\left(\sqrt{2},\sqrt{3}\right)$	$Q\left(\sqrt{2+\sqrt{2}}\right)$	$\frac{\sqrt{\sqrt{2}+1}}{2}$

Table 1: Algebraic fields associated with triangles of Group I

Therefore the arithmetic types of the orbit traces are obtained from those of its rotational part by multiplication by ρ; again we can have at most four additional types of the length multiplets; they all belong to the set $\rho \times Q_T$. It follows that the eo-trace degeneracy and the Artin-like doublet structure in the PO length spectrum can exist only if ρ belongs to Q_T. E. g., for Artin's billiard $(2,3,\infty)$ the field Q_T coincides with the field of rational numbers Q which contains $\rho = 1/2$. On the other hand, for the billiard $(2,3,8)$ we have

$$Q_T = Q\left(\cos\frac{\pi}{8}\right) = Q\left(\sqrt{2+\sqrt{2}}\right),$$

$$\rho = \sqrt{\cos^2\frac{\pi}{8}+\cos^2\frac{\pi}{3}-1} = \frac{1}{2}\sqrt{\sqrt{2}-1} \notin Q_T$$

Correspondingly the doublet structure is present in the first example and absent in the second one.

4 Classification of arithmetic triangles

Next we study the arithmetic triangles grouping them according to the three scenarios mentioned; the number of types refers to orbits with fixed parity of ν.

4.1 Group I: Four trace types; triangles $(2,n,m)$ with m,n even and n non-divisible by m [4 systems]

The group consists of the right triangles $(2,4,6),(2,4,10),(2,4,18),(2,6,8)$. Table I contains the fields encountered in the two approaches for these systems. In all cases the fields K, K' do not coincide; $\cos\alpha$ does not belong to K and $\cos\beta$ does not belong to K'; that gives the division of the field K_T into four non-overlapping subsets. Consequently the length multiplets can be of four types depending on parity of ν_M and ν_N; parity of ν_L is fixed by the condition that the overall number of reflections ν is even or odd. Within a degenerate length multiplet parity of all three numbers ν_L,ν_M,ν_N is fixed, hence all orbits of the multiplet acquire the same reflection phase modulo 2π. The quantum energy spectrum is thus always be Poissonian and the triangle is genuinely arithmetic regardless of the boundary conditions.

Let us look in more detail, e. g., at the triangle $(2, 4, 10)$ starting with ν even; denote $\kappa = 2\cos \pi/10 = \sqrt{\frac{5+\sqrt{5}}{2}}$. If ν_M is even the orbit traces will have the structure $q\left(\sqrt{5}, \sqrt{2}\right) = q_0 + q_1\sqrt{2} + \sqrt{5}\left(q_2 + q_3\sqrt{2}\right)$ where $q_{0\ldots 3}$ are rationals; with ν_M odd they are $\kappa\, q\left(\sqrt{5}, \sqrt{2}\right)$. In the mirror approach the traces have structure $q\left(\kappa\right) = q_0' + q_1'\kappa + \sqrt{5}\left(q_2' + q_3'\kappa\right)$ if ν_N is even and $\sqrt{2}q\left(\kappa\right)$ if ν_N is odd. Now, when both ν_M and ν_N are even (and consequently ν_L is also even) the trace is simultaneously $q\left(\sqrt{5}, \sqrt{2}\right)$ and $q\left(\kappa\right)$, i.e., must be $q\left(\sqrt{5}\right) = q_0 + q_1\sqrt{5}$. The traces structure for other combinations of parities of $\nu_{M/N}$ for ν even is given in Table 2 (e=even, o=odd); these results can be seen as a refinement of the general formula (6). For the orbits with ν odd the additional common factor $\rho = \sqrt{\frac{\sqrt{5}+1}{8}}$ appears; remembering that the rotational body of the code contains $\nu_M - 1$ reflections (resp. $\nu_N - 1$ reflections in the mirror approach) we get an analogous Table 3. It can be checked that ρ does not belong either to Q or to Q', hence there is no Artin's-like approximate *eo* orbit length degeneracy.

ν_M	ν_N	ν_L	Trace
e	e	e	$q\left(\sqrt{5}\right)$
e	o	o	$q\left(\sqrt{5}\right)\sqrt{2}$
o	e	o	$q\left(\sqrt{5}\right)\kappa$
o	o	e	$q\left(\sqrt{5}\right)\kappa\sqrt{2}$

Table 2: Triangle $(2, 4, 10)$, even ν

ν_M	ν_N	ν_L	Trace
o	o	o	$q\left(\sqrt{5}\right)\rho$
o	e	e	$q\left(\sqrt{5}\right)\rho\sqrt{2}$
e	o	e	$q\left(\sqrt{5}\right)\rho\kappa$
e	e	o	$q\left(\sqrt{5}\right)\rho\kappa\sqrt{2}$

Table 3: Triangle $(2, 4, 10)$, odd ν

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4.2   Group II: Two algebraic types [51 systems]

This is the largest and fairly heterogeneous group.

4.2.1   Non-symmetric right triangles $(2, m, n)$ with m odd or infinite, n even

These are the triangles $(2, 3, n)$, $n = 8, 10, 12, 14, 16, 18, 24, 30$; $(2, 5, n)$, $n = 4, 6, 8, 10, 20, 30$; $(2, 7, 4)$, $(2, 7, 14)$, $(2, 9, 18)$, $(2, 15, 30)$, $(2, \infty, 4)$, $(2, \infty, 6)$.

These triangles have properties similar to the previously investigated $(2, 3, 8)$. For all of them $\cos \alpha \notin Q\left(\cos 2\alpha, \cos \beta\right)$ but $\cos \beta \in Q\left(\cos \alpha, \cos 2\beta\right)$. One can easily check it with the help of (2) considering that $\alpha = \pi/n$ with n even and $\beta = \pi/m$ with m odd. It follows that the trace field K_T is divided into the subfield $K = Q\left(\cos 2\alpha, \cos \beta\right)$ and its "coset" $\cos \alpha\, K$; the mirror approach doesn't produce an alternative division of K_T. The traces of orbits with an even total number of reflections ν belong thus either to K or to $\cos \alpha\, K$ when ν_M is even and odd, respectively. Therefore the length multiplets cannot contain orbits with different parity of ν_M and $\nu_L + \nu_N$; individual parities of ν_L, ν_N are not fixed. The reflection phases of orbits within the length multiplets will

be equal only if the boundary conditions on the hypotenuse L and the leg N adjacent to the angle β coincide; only then the quantum level statistics will be Poissonian and the triangle genuinely arithmetic.

For the orbits with odd ν the traces belong to $\rho \cos \alpha Q (\cos 2\alpha, \cos \beta)$ or $\rho Q (\cos 2\alpha, \cos \beta)$, when ν_M is even or odd, respectively; conclusions on parity of the number of reflections of the orbits in the length multiplets and the quantum level statistics remain the same.

As an example, we show in Fig. 4a the periodic orbit lengths of the triangle with the angles $(\pi/2, \pi/3, \pi/n)$, $7.95 < n < 8.05$. The colors indicate one of the four possible combinations of parities of the reflection numbers ν and ν_M. At $n = 8$ the triangle becomes arithmetic $(2, 3, 8)$ which is indicated by simultaneous multiple crossings in the plot. Note "the color segregation": only lines of the same color (=same parity of ν, ν_M) are allowed to cross at $n = 8$. Fig.4b shows the blow-up in the length interval $11.011 < l < 11.015$.
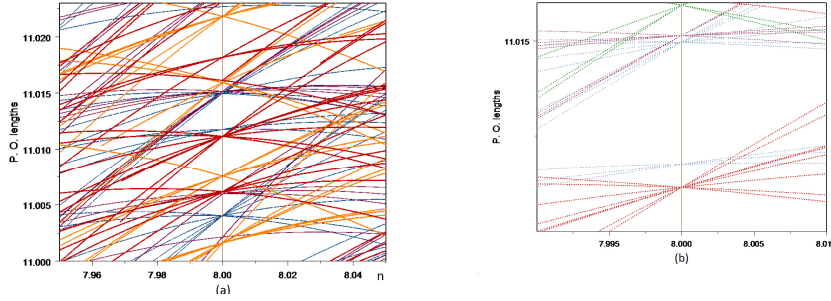

Figure 4: a) Stretch of PO length spectrum of the triangle $(2, 3, n)$ with n close to 8. The color (red, blue, yellow or magenta) indicates combination of parities of the reflection numbers ν, ν_L. Only lines of the same color cross at the arithmetic point $n = 8$. b) Blow-up of the length interval $11.011 < l < 11.015$

4.2.2 Right triangles $(2, 2k, 4k)$, $k = 2, 3, 4, 6$

Here $\beta = 2\alpha = \pi/2k$. Again we have $\cos \alpha \notin Q (\cos 2\alpha, \cos \beta) = Q (2\alpha)$ such that the orbits with even and odd number of reflections ν_M from the side opposite to β cannot have equal length; the mirror approach doesn't produce an alternative division since $\cos \beta = \cos 2\alpha \in Q (\cos \alpha, \cos 2\beta) = Q (\cos \alpha) = K_T$. The statements of 4.2.1 can now be repeated: parities of ν_M and $\nu_L + \nu_N$, not ν_L, ν_N separately, are fixed within the length multiplets. The quantum triangle is genuinely arithmetic iff the boundary conditions on the hypotenuse and the leg adjacent to $\beta = \pi/2k$ are the same; we don't know whether that result was previously reported.

4.2.3 Triangle $(2, 4, 12)$

That right triangle is singled out because the method used up to now fails to explain the numerically observed division of traces into two types. Indeed according to the first approach in the case of even ν the orbit traces belong to the field $K = Q\left(\cos \pi/6, \cos \pi/4\right)$ or to $K \times \cos \pi/12$ depending on parity of ν_M. However $\cos \pi/12 = \frac{\sqrt{2}+\sqrt{6}}{4} \in K$ such that K and $K \cos \pi/12$ coincide with each other and with the full field K_T. It follows that parity of ν_M in the length multiplets is not arithmetically fixed. The mirror approach is similarly unproductive because $K' = Q\left(\cos \pi/12\right) = K_T$; consequently parity of ν_N is also not fixed.

In fact, the roles are changed in that triangle: it is parities of ν_L and $\nu_M + \nu_N$ which are fixed in the multiplets such that the quantum triangle is genuinely arithmetic if the legs M and N have the same boundary conditions; the boundary condition on the hypothenuse L is arbitrary. Details can be found in Appendix A. That is unique among the non-symmetric right triangles and probably connected with the fact that the lengths of the legs M, N are in $2 : 1$ relation.

4.2.4 Symmetric triangles whose two equal angles are even fractions of π

These include 5 right triangles $(2, 2k, 2k), k = 3 - 6, 9$, and 16 acute ones: $(3, 2k, 2k), \quad k = 2, 3, 4, 6; \quad (4, 2k, 2k), \quad k = 3, 4, 8; \quad (5, 4, 4), (5, 10, 10);$ $(6, 4, 4)), (6, 12, 12), (6, 24, 24); \quad (9, 4, 4), (9, 18, 18); \quad (\infty, 4, 4), (\infty, 6, 6).$

These triangles can be desymmetrized by introducing an artificial wall along the line of symmetry. Periodic orbits of the full triangle can be folded into its half ("the fundamental domain") with the mirror reflection from the artificial wall. The result is a right arithmetic triangle; if it is non-symmetric it belongs to one of the types studied above; if it is still symmetric, one more step of desymmetrization is necessary. It is important that not all orbits in the desymmetrized triangle but only those which have even number of reflections from the fictitious wall, correspond to the orbits in the original symmetric triangle.

The detailed description is given in Appendix B. Here we mention that if the desymmetrized triangle belongs to the Group I, only two of the four possible algebraic structures of the trace are realized; therefore the length spectrum is halved compared with the desymmetrized triangle. On the other hand, if the desymmetrization result belongs to the Group II the length spectrum of the symmetric and the desymmetrized triangle coincide; what differs is the multiplicities in the spectra. In both cases the periodic orbits within a length multiplet of the original symmetric triangle have fixed parity of the total number of reflections from the equal sides, not from each of them separately. Hence the reflection phase in the Gutzwiller expansion has the same value, modulo 2π, for all orbits within the multiplets iff the boundary conditions on the symmetric walls coincide; the quantum triangle is then genuinely arithmetic.

4.2.5 Non-symmetric acute triangles

These are $(3, 4, 6)$, $(3, 4, 12)$, $(3, 6, 18)$, $(3, 8, 24)$, $(3, 10, 30)$. Here we had to calculate anew the elementary matrices of reflection and rotation; investigation and detailed results are given in the Appendix C. The conclusion is that the length multiplets have fixed parity of the number of reflections from the side L opposite to the angle $\pi/3$ whereas parities of ν_M and ν_N are not fixed. Therefore the triangles are genuinely arithmetic if the boundary conditions at the sides adjacent to $\pi/3$ are the same; that coincides with the group theoretical prediction mentioned in the Introduction.

4.3 Group III. Single algebraic type [30 systems]

The group includes,

- 9 equilateral triangles (k, k, k), $k = 4 - 9, 12, 15, \infty$;

- 21 non-equilateral triangles (l, m, n) in which two or more of the numbers l, m, n are odd or infinite. These are the right triangles $(2, 3, 7)$, $(2, 3, 9)$, $(2, 3, 11)$, $(2, 3, \infty)$, $(2, 5, 5)$, $(2, 7, 7)$, $(2, \infty, \infty)$ and the acute ones $(3, 3, k)$, $k = 4 - 9, 12, 15, \infty$; $(3, 5, 5)$, $(3, \infty, \infty)$, $(4, 5, 5)$, $(5, 5, 10)$, $(5, 5, 15)$

In these triangles parity of neither ν_L nor ν_M nor ν_N is fixed within the length multiplet, only that of their sum ν. The quantum triangle is genuinely arithmetic only if the boundary conditions on all sides are the same. That conclusion can be confirmed as earlier; e. g., the equilateral triangles can be desymmetrized resulting in the triangle $(2, k, 2k)$ with $\alpha = \pi/2k$, $\quad \beta = \pi/k$. The wall M adjacent to α would be a fictitious one such that the number of reflections ν_M of the orbit folded into the fundamental domain must be even. It was shown above that the orbit length spectrum of the triangles $(2, k, 2k)$ is divided into subspectra with respect to parity of ν_M. However, since odd ν_M are not allowed now, only the division of the orbit multiplets with respect to parity of ν remains.

The triangles containing two angles which are odd fractions of π are treated by desymmetrization, if needed, and the usage of (2).

5 Conclusion

Depending on boundary conditions, triangular billiards on the pseudosphere with one of the 85 "arithmetic" sets of angles, can have either Poissonian statistics of its energy levels, in spite of its completely chaotic classical dynamics, or conform to GOE. From the semiclassical point of view, the peculiar properties of arithmetic systems result from constructive interference of contributions of an abnormally large number of periodic orbits with exactly the same length and action. In fact, not only the length but also the Maslov phase of the orbits needs to be equal; for a billiard on the pseudosphere that means that orbits within

every length multiplet must have the same total phase gained in reflections from the sides with Dirichlet boundary condition. Coincidence of the reflection phases can occur only because of special properties of the periodic orbits of arithmetic billiards. These properties were the topic of this paper. One such property is that only orbits with the same parity of the total number of reflections can have the same length. That rule holds, with statistically insignificant exceptions, for all arithmetical triangles and guarantees that billiards with all-Dirichlet sides are arithmetic, never pseudo-arithmetic. Other similar properties of periodic orbits depend on the triangle in question and concern the parity of the individual number of reflections from the billiard sides. These properties have been investigated for all arithmetic trangles and boundary conditions have been established under which the billiards are arithmetic and pseudo-arithmetic.

The arithmetic/pseudo-arithmetic division of triangles based on the periodic orbit analysis must of course give the same results as group-theoretical considerations. Indeed, the well-known rule of the latter approach that the sides of a genuinely arithmetic triangle adjacent to the angle π/k with k odd must have the same boundary conditions [17, 16], is confirmed by the results of the present paper. The equivalence of the two approaches is almost self-evident in the case of symmetric triangles. A group-theoretical derivation of the boundary conditions of arithmeticity for the few remaining triangles can probably be obtained, too.

Our conclusions on the spectral statistics were based on the diagonal approximation. However, the distinction between the arithmetic and pseudo-arithmetic cases must survive even if higher-order terms are taken into account. The standard off-diagonal contributions of the GOE class stem from pairs of orbits consisting of the same pieces, some of them time-reversed, connected in different order and therefore having the same reflection phase [6, 7], and only these contributions survive in the pseudo-arithmetic case. In the arithmetic case the set of contributing pairs must be much more diverse including pairs of orbits with close action but otherwise unrelated; the problem has not been investigated so far. Specific off-diagonal effects are expected in systems like Artin's billiard where exact equality is allowed of the matrix traces associated with the orbits with even and odd number of reflections leading to the doublet structure of the length spectrum; the form factor experiences then a crossover at the Ehrenfest time..

6   Acknowledgement

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A   Triangle $(2, 4, 12)$

We begin with the even total number of reflections ν when the orbit matrix can be represented as a product of the elementary rotation matrices. The orbit matrices turn out to be of two types,

$$R^I = \begin{pmatrix} u_1 + u_2 3^{1/4}\sqrt{6} & u_3 + u_4\ 3^{1/4}\sqrt{6} \\ -u_3 + u_4\ 3^{1/4}\sqrt{6} & u_1 - u_2 3^{1/4}\sqrt{6} \end{pmatrix}$$

and

$$R^{II} = \begin{pmatrix} u_1\sqrt{2} + u_2 3^{1/4} & u_3\sqrt{2} + u_4\ 3^{1/4} \\ -u_3\sqrt{2} + u_4 3^{1/4} & u_1\sqrt{2} - u_2 3^{1/4} \end{pmatrix}$$

where $u_{1/2/3/4}$ denote algebraic numbers belonging to the field $Q\left(\sqrt{3}\right)$, i.e., $u = q_0 + q_1\sqrt{3}$. The multiplication rules for the product of two matrices can be symbolically written,

$$R^I R^I = R^I, \quad R^I R^{II} = R^{II}, \quad R^{II} R^I = R^{II}, \quad R^{II} R^{II} = R^I$$

It can be directly checked using (5) with $\alpha = \pi/12, \beta = \pi/4$ that the rotation matrix $R_Q = \Sigma_M \Sigma_N$ belongs to the type R^I whereas $R_P = \Sigma_N \Sigma_L$ and $R_O = \Sigma_L \Sigma_M$ belongs to R^{II}. It follows that the type of the orbit matrix is defined by the total number of elementary rotations R_P and R_O which is equal to the number of reflections ν_L from the hypotenuse L. Consequently orbits in the length multiplets have definite parity of ν_L and $\nu_M + \nu_N$; the algebraic
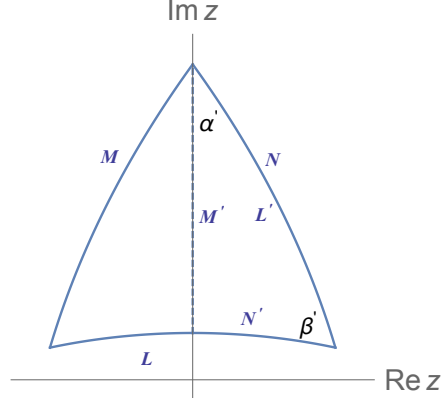
Figure 5: Desymmetrization of a triangle with equal sides

structure of the orbit traces is $q_0 + q_1\sqrt{3}$ or $q_0\sqrt{2} + q_1\sqrt{6}$ when ν_L is even or odd, respectively. The quantum triangle will be genuinely arithmetic if the boundary conditions on the legs M, N coincide.

In the case of odd ν we represent the orbit code as $\Sigma_M R$ and obtain that $\mathrm{Tr}\,\Sigma_M R = R_{11} - R_{22}$ has the algebraic structure $\left(q_0\sqrt{2} + q_1\sqrt{6}\right) 3^{1/4}$ if ν_N is even and $\left(q_0 + q_1\sqrt{3}\right) 3^{1/4}$ if ν_L is odd.

B   Triangles with two equal angles which are even fractions of π

In Fig. 3 we show a symmetric triangle whose symmetry axis is directed along the imaginary axis in the Poincaré half-plane,

Desymmetrizing the problem we draw a fictitious wall denoted M' along the line of symmetry; using notations of our first approach we denote the side N in the full triangle by L' in the desymmetrized one, and half of the side L by N'. Then,

- the number ν'_N of reflections from N' in the folded orbit coincides with ν_L in the original one;

- the side L' in the desymmetrized triangle collects reflections from the sides M and N of the full triangle such that $\nu'_L = \nu_M + \nu_N$;

- the number of reflections ν'_M is even.

Desymmetrizing $(l; 2k, 2k)$ we obtain the right triangle $(2, m', n')$ with even $m' = 2l$ and $n' = 2k$. Unless $n = k$ it will be non-symmetric and belong to one

of the considered types. The algebraic limitations on parity of the number of reflection concern $\nu'_L = \nu_M + \nu_N$, not ν_M, ν_N separately, such that the symmetric sides M and N must have the same boundary conditions for the quantum triangle to be genuinely arithmetic; this wass of course to be expected.

Now we give a survey of the results.

A) In 8 triangles a single step of desymmetrization produces a triangle of the Group I. The matrix traces in that group can be of four types, however only two of them survive since the number of strikes against the fictitious wall must be even.

Original $(l, 2k, 2k)$	Desymmetrized $(2, 2l, 2k)$
$(2, 6, 6)$	$(2, 4, 6)$
$(2, 10, 10)$	$(2, 4, 10)$
$(2, 18, 18)$	$(2, 4, 18)$
$(3, 4, 4)$	$(2, 6, 4)$
$(3, 8, 8)$	$(2, 6, 8)$
$(4, 6, 6)$	$(2, 8, 6)$
$(5, 4, 4)$	$(2, 10, 4)$
$(9, 4, 4)$	$(2, 18, 4)$

Using our results on Group I it is easy to show that for orbits with even $\nu = \nu_N + \nu_L + \nu_M$ the traces belong to the field $K = Q\left(\cos \pi/l, \cos \pi/k\right)$ if ν_L is even, and to $K \cos \pi/k$ if ν_L is odd. With odd ν the traces belong to $\rho\left(\pi/2n, \pi/2k\right) K \cos \pi/k$ and $\rho\left(\pi/2n, \pi/2k\right) K$ if ν_L is even and odd, respectively.

B) In 6 cases the desymmetrized triangle belongs to the Group II:

Original $(l, 2k, 2k)$	Desymmetrized $(2, 2l, 2k)$
$(2, 8, 8)$	$(2, 4, 8)$
$(2, 12, 12)$	$(2, 4, 12)$
$(3, 12, 12)$	$(2, 6, 12)$
$(4, 8, 8)$	$(2, 8, 16)$
$(6, 4, 4)$	$(2, 4, 12)$
$(6, 24, 24)$	$(2, 12, 24)$
$(\infty, 4, 4)$	$(2, \infty, 4)$
$(\infty, 6, 6)$	$(2, \infty, 6)$

Connection of parity of ν_L and the algebraic properties of the traces is the same as in A). Note the difference: half of lengths in the orbit length spectrum of the symmetric triangle of the case A) disappears compared with the corresponding desymmetrized triangle since two of the four trace algebraic types are not allowed for the orbits obtained by folding. On the other hand, in the case B) the length spectra of the symmetrized and desymmetrized systems coincide, only the multiplicities in the spectra differ.

D) The 5 remaining symmetric triangles need one more stage of desymmetrization

Original $(l; 2k, 2k)$	Once desymmetrized	Second stage, $(2, 4, 2k)$
$(3, 6, 6)$	$(2, 6, 6)$	$(2, 4, 6)$
$(4, 8, 8)$	$(2, 8, 8)$	$(2, 4, 8)$
$(5, 10, 10)$	$(2, 10, 10)$	$(2, 4, 10)$
$(6, 12, 12)$	$(2, 12, 12)$	$(2, 4, 12)$
$(9, 18, 18)$	$(2, 18, 18)$	$(2, 4, 18)$

Denoting the sides of the fundamental domain $(2, 4, 2k)$ as L'', M'', N'' where the new fictitious wall N'' is the bisector of the right angle of the intermediate triangle $L'M'N'$ and the side L'' coincides with N', we shall have $\alpha'' = \angle L''M'' = \pi/2k$, $\beta'' = \angle N''L'' = \pi/4$.The number of bounces $\nu_{L''}$ of the double-folded orbit against L'' is the sum $\nu_{M'} + \nu_{N'} = \nu_{M'} + \nu_L$; since $\nu_{M'}$ must be even, parity of $\nu_{L''}$ and ν_L is the same. The number of bounces $\nu_{M''}$ from L'' coincides with $\nu'_N = \nu_N + \nu_M$, i.e. has the same parity as ν_L. The number of bounces $\nu_{N''}$ from the second fictitious wall N'' is of course even. We can conclude that for orbits with even ν the orbits traces belongs to $K = Q(\cos \pi/k)$ if ν_L is even, and to $\cos \pi/2k\, Q(\cos \pi/k)$ if ν_L is odd. If ν is odd the factor $\rho = \sqrt{\cos^2 \frac{\pi}{2k} - \frac{1}{2}}$ is to be added and parities of ν_L interchanged.

C   Non-symmetric acute triangles

We place the triangle in the Poincaré plane such that the side $M = OQ$ is directed along the imaginary axis, the vertex Q at the angle $\pi/3$ has the coordinate $y_Q = 1$. The vertex O at the angle $\alpha = \pi/n$ has then the coordinate

$$y_O = \frac{1}{\sqrt{3}\sin \alpha} (\cos \alpha + 2 \cos \beta + \rho),$$
$$\rho(\alpha, \beta) \equiv \sqrt{1 + 2 \cos 2\alpha + 2 \cos 2\beta + 4 \cos \alpha \cos \beta};$$

the vertex P at the angle $\beta = \pi/m$ is chosen to lie in the first quadrant of the Poincaré plane. We denote by $\Sigma_{L/M/N}$ the matrices of reflections, in particular, as previously $\Sigma_M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.The matrices of rotation about the vertices read,

$$R_O = \Sigma_M \Sigma_L = \begin{pmatrix} \cos \alpha & -\frac{1}{\sqrt{2}}(\cos \alpha + 2 \cos \beta + \rho) \\ \frac{1}{\sqrt{2}}(\cos \alpha + 2 \cos \beta - \rho) & \cos \alpha \end{pmatrix},$$
$$R_Q = \Sigma_N \Sigma_M = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix};$$

the matrix of rotation $R_P = \Sigma_N \Sigma_L$ can be replaced by $R_Q R_O$. Any code with even number of reflections ν, can be written as a sequence of R_O and R_Q.

It will be convenient to denote the triangles here as
$(l, m, n) = (3, 6, 4), (3, 12, 4), (3, 18, 4), (3, 24, 8), (3, 30, 10)$. The first triangle stands apart and will be treated separately.

C.1 Triangle $(3, 6, 4)$

Here the logic follows the case of $(2, 3, 8)$. Substituting $\alpha = \pi/4, \beta = \pi/6$ we obtain,

$$R_O = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\left(6+\sqrt{6}+2r\sqrt{3}\right)}{6} \\ \frac{6+\sqrt{6}-2r\sqrt{3}}{6} & \frac{\sqrt{2}}{2} \end{pmatrix},$$

$$r = \rho\left(\pi/4, \pi/6\right) = \sqrt{2 + \sqrt{6}}.$$

The orbit matrices R are divided into two types which can be written in terms of the algebraic numbers

$$u = q_0 + q_1\sqrt{6} \in Q\left(\sqrt{6}\right),$$

$$v = q_0\sqrt{2} + q_1\sqrt{3} \in \sqrt{2}Q\left(\sqrt{6}\right)$$

where $q_{0/1}$ are rationals, as

$$R^I = \begin{pmatrix} u_1 + v_1 r & v_2 + u_2 r \\ -v_2 + u_2 \ r & u_1 - v_1 r \end{pmatrix}$$

and

$$R^{II} = \begin{pmatrix} v_1 + u_1 r & u_2 + v_2 r \\ -u_2 + v_2 \ r & v_1 - u_1 r \end{pmatrix}.$$

It is easily checked that multiplication by a R^{II}-matrix changes the type of the matrix whereas that by R^I doesn't; symbolically, $R^I R^I = R^I$, $R^I R^{II} = R^{II}$, $R^{II} R^{II} = R^I$. The elementary rotation R_Q belongs to the type I with $u_1 = -1/2$, $v_2 = \sqrt{3}/2$, $v_1 = u_2 = 0$; the rotation R_O belongs to the type II with $v_1 = \sqrt{2}/2$, $u_1 = 0$, $u_2 = -1 - \sqrt{6}/6$, $v_2 = -\sqrt{3}/3$. It follows that the orbit matrix belongs to the type I (resp. II) if the orbit code contains an even (resp. odd) number of elementary rotations R_O; the number of rotations R_Q is irrelevant.

Turning to the matrix traces and reformulating the result in terms of the number of reflections $\nu_{L/M/N}$ we get that the trace is given by $u \in Q\left(\sqrt{6}\right)$ (resp. $v \in \sqrt{2}Q\left(\sqrt{6}\right)$) for ν_L even (resp. odd). We got thus a refinement of the general formula (6) which would give $\text{Tr}\,R \in Q\left(\sqrt{2}, \sqrt{3}\right)$. Since non-zero u and v cannot be equal, the orbit length spectrum falls into two subspectra; the orbits within each degenerate multiplet have the same parity of ν_L and $\nu_M + \nu_N$ but not of ν_M, ν_N individually. Consequently, in the quantum problem the statistics will be Poissonian and the triangle genuinely arithmetic iff the boundary conditions at the sides M, N are the same (both Neumann or both Dirichlet).

In the case of odd ν , i.e, the inverse hyperbolic orbits, we represent the orbit code as $\Sigma_M \mathbb{R}$ and obtain the trace of the orbit matrix as the difference of the diagonal elements of the rotational body. It follows that the trace belongs to $rQ\left(\sqrt{6}\right)$ if ν_L is even, otherwise it is $r \sqrt{2}Q\left(\sqrt{6}\right)$ These two sets don't have

common non-zero elements such that $\nu_L, \nu_M + \nu_N$ have fixed parity within the length multiplets; all conclusions for the even ν remain thus in force. Note that $r = \sqrt{2 + \sqrt{6}}$ does not belong to $Q\left(\sqrt{2}, \sqrt{3}\right)$, i.e., traces in the odd case belong to an extension of the field K_T.

C.2 Triangles $(3, 12, 4), (3, 18, 6), (3, 24, 8), (3, 30, 10)$

In the remaining non-right triangles we have $m = 3n$ such that $\alpha = 3\beta$. We start with the case of even ν when the code is a product of elementary rotations. The matrices of the orbit code fall into two types; introducing $y = 2\cos\beta$ we can write them as

$$
R^I = \begin{pmatrix} \frac{P_1^I\left(y^2\right) + \sqrt{y^2 - 3}Q_1^I\left(y^2\right)}{2} & \frac{P_2^I\left(y^2\right) + y\sqrt{y^2 - 3}Q_2^I\left(y^2\right)}{2\sqrt{3}} \\ \frac{-P_2^I\left(y^2\right) + y\sqrt{y^2 - 3}Q_2^I\left(y^2\right)}{2\sqrt{3}} & \frac{P_1^I\left(y^2\right) - \sqrt{y^2 - 3}Q_1^I\left(y^2\right)}{2} \end{pmatrix} \tag{11}
$$

and

$$
R^{II} = \begin{pmatrix} \frac{yP_1^{II}\left(y^2\right) + \sqrt{y^2 - 3}Q_1^{II}\left(y^2\right)}{2} & \frac{yP_2^{II}\left(y^2\right) + \sqrt{y^2 - 3}Q_2^{II}\left(y^2\right)}{2\sqrt{3}} \\ \frac{-yP_2^{II}\left(y^2\right) + \sqrt{y^2 - 3}Q_2^{II}\left(y^2\right)}{2\sqrt{3}} & \frac{yP_1^{II}\left(y^2\right) - \sqrt{y^2 - 3}Q_1^{II}\left(y^2\right)}{2} \end{pmatrix}. \tag{12}
$$

Here $P_{1,2}^{I,II}\left(y^2\right), Q_{1,2}^{I,II}\left(y^2\right)$ are polynomials of y^2 with rational coefficients. Multiplication rules for matrices belonging to the two types are the same as in the preceding section, $R^I R^I = R^I, \quad R^I R^{II} = R^{II}, \quad R^{II} R^{II} = R^I$. The elementary rotation matrix R_Q belongs to the type I with $P_1^I = -1, \quad P_2^I = 3, \quad Q_I^1 = Q_2^I = 0$; the matrix R_O is of the type II with $P_1^{II} = y^2 - 3, \quad Q_1^{II} = 0, \quad P_2^{II} = 1 - y^2, \quad Q_2^{II} = 2\left(1 - y^2\right)$.

Using the multiplication rules we can prove (11),(12) and obtain that the code matrix belongs to the type I (resp. II) if the code contains an even (resp. odd) number of rotations R_O equal to the number of reflections ν_L . The matrix traces are given by $P_1^I\left(y^2\right) \in Q\left(\cos 2\beta\right)$ or $yP_1^{II}\left(y^2\right) \in \cos\beta\, Q\left(\cos 2\beta\right)$, respectively; since $\beta = \pi/m$ with m even, these two sets do not have non-zero common elements, i. e., the orbits with ν_L of different parity cannot have equal lengths. Consequently, parity of ν_L, same as that of $\nu_M + \nu_N$, is fixed within each length multiplet whereas individual parities of ν_M, ν_N are not. Therefore the quantum triangle is genuinely arithmetic iff the boundary conditions on M and N are both Dirichlet or both Neumann.

The case of odd ν is treated similar to $(3, 6, 4)$; the traces belong to $\rho Q\left(\cos 2\beta\right)$ or to $\rho Q\left(\cos 2\beta\right)\, \cos\beta$ with $\rho = \sqrt{4\cos^2\beta - 3}$ if ν_L is even or odd, respectively.